# DAM-BREAK FLOWS OVER A BOTTOM STEP

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*A single-layer shallow-water model is used to study the solvability of the problem of flows generated by dam break over a bed level discontinuity in the form of a step onto which water flows. Solutions in which the total flow energy is conserved on the step and solutions in which the energy is lost on the step are considered.*

**Key words:** *shallow water, dam break, bottom step.*

**1. Formulation of the Problem.** In the case of a rectangular channel of constant width and variable depth, the differential single-layer shallow-water equations (Saint Venant equations) [1, 2] ignoring the friction effect are written as

$$h_t + q_x = 0, \qquad q_t + (qv + h^2/2)_x = -hb_x, \tag{1.1}$$

where $h(x,t)$, $q(x,t)$, and $v = q/h$ are the flow depth, rate, and velocity and $b(x)$ is the bed level. The acceleration of gravity $g = 1$. For system (1.1), we consider the problem of decay of an initial level discontinuity $z = b + h$

$$z(x,0) = \begin{cases} z_0, & x > 0, \\ z_1, & x < 0, \end{cases} \qquad z_1 > z_0 \tag{1.2}$$

over a sudden change in bed level

$$b(x) = \begin{cases} \delta, & x > 0, \\ 0, & x < 0, \end{cases} \qquad \delta > 0 \tag{1.3}$$

in water initially at rest:

$$v(x,0) = 0. \tag{1.4}$$

Because $z_1 > z_0$, it follows that $q(0,t) > 0$ at $t > 0$ and in the nomenclature adopted in [3], the discontinuity (1.3) is a bottom step on which water flows. In addition, taking into account that the problem of discontinuity decay (1.1), (1.2), (1.4) over an even bottom is called the dam break problem [4], we shall call problem (1.1)–(1.4) the problem of dam break over a bottom step. Furthermore, the exact solution at the discontinuity (1.3) at the point $x = 0 - 0$ will be called flow ahead of the step, and that at the point $x = 0 + 0$ will be called flow at the step. The solution at $x < 0$ will be called flow on the left of the step, and the solution at $x > 0$, flow on the right of the step.

Problem (1.1)–(1.4) is a particular case of the general problem of arbitrary discontinuity decay over a sudden change in bed level, which was studied in [5], where various examples of its solution are given assuming that at the discontinuity (1.3), the total flow energy is conserved. However, in [5], the uniqueness of these solutions was not studied and the regions of their existence were not distinguished. Problem (1.1)–(1.4) with $\delta < 0$ for the case of a bed level discontinuity in the form of a step from which water flows down was investigated in [6] within the framework of theoretical analysis of hydraulic processes arising from break of a flight ship lock-gate. Problem (1.1)–(1.4) was also studied in [7] using the assumption that the total flow energy at the discontinuity (1.3) is conserved. However, in [7], the one-valued solvability of this problem was shown only approximately by numerical calculations, which indicate strict monotonicity of the corresponding functional dependence.

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The goal of the present work is to analyze the one-valued solvability of the discontinuity decay problem (1.1)–(1.4) assuming both that the total flow energy is conserved after passage through the bottom step and that it is lost at it. In the analysis, we use the results obtained in studies of discontinuity decay in shallow water over a horizontal bottom [8], gas-dynamic discontinuity decay at a sudden change in pipe cross-sectional area [9, 10], and permissible steady flows at the discontinuity (1.3) [7].

Before solving problem (1.1)–(1.4), we recall how to prove the one-valued solvability of the classical discontinuity decay problem over a horizontal bottom $b(x) = \text{const}$:

$$h(x,0) = \begin{cases} h_0, & x > 0, \\ h_1, & x < 0, \end{cases} \qquad v(x,0) = \begin{cases} v_0, & x > 0, \\ v_1, & x < 0. \end{cases} \tag{1.5}$$

**2. Solution of the Classical Problem of Arbitrary Discontinuity Decay.** In [8], the arbitrary discontinuity decay problem (1.1), (1.5) for $b_x = 0$ is solved by analogy with the gas-dynamic case [11] using $s$- and $r$-adiabats. The $s$-adiabat passing through initial state $h_0, v_0$ is a function

$$v = v_s(h, h_0, v_0) = v_0 + a(h, h_0), \tag{2.1}$$

and the $r$-adiabat passing through initial state $h_1, v_1$ is a function

$$v = v_r(h, h_1, v_1) = v_1 - a(h, h_1), \tag{2.2}$$

where

$$a(h, h_i) = \begin{cases} \sqrt{(h+h_i)/(2hh_i)}\,(h - h_i), & h \geqslant h_i, \\ 2(\sqrt{h} - \sqrt{h_i}\,), & h \leqslant h_i. \end{cases} \tag{2.3}$$

For $h > h_i$, the equations of the adiabats (2.1)–(2.3) obtained from the Hugoniot conditions

$$D[h] = [q], \qquad D[q] = [qv + h^2/2] \tag{2.4}$$

($D$ is the shock-wave velocity and $[f]$ is a discontinuity of the function $f$ at the shock-wave front) relate the initial values of $h_i$, and $v_i$ ahead of a discontinuous wave front to the possible states $h, v$ behind its front. For $h < h_i$, these equations, obtained from the conditions

$$s = v + 2c = \text{const}, \qquad r = v - 2c = \text{const} \tag{2.5}$$

($c = \sqrt{h}$ is the speed of propagation of small perturbations in stationary water) relate the initial values of $h_i$ and $v_i$ ahead of a centered depression wave to the possible states $h, v$ behind it. The Hugoniot conditions (2.4) are obtained from the laws of conservation of mass and total momentum (1.1), and conditions (2.5) follow from the constancy of the $s$-invariant in an $r$-depression wave and constancy of the $r$-invariant in an $s$-depression wave [11].

The one-valued solvability of the classical discontinuity decay problem (1.1), (1.5) follows from a monotonic increase in the $s$-adiabat (2.1) and a monotonic decrease in the $r$-adiabat (2.2) (see [8]), which generally results in the formation of a simple $s$-wave propagating over background $h_0, v_0$ and a simple $r$-wave propagating over background $h_1, v_1$; this wave are joined by a constant flow region. In particular, in the solution of the classical problem of dam-break flow (1.1), (1.2), (1.4) over a horizontal bottom $b(x) = \text{const}$, a discontinuous $s$-wave propagates over background $h_0, v_0$ and a centered $r$-depression wave propagates over background $h_1, v_1$.

In the solution of the generalized problem of discontinuity decay over a step (1.1)–(1.4), the resulting flow pattern is more complicated but, as shown in [10], the graphical method of adiabats is also effective in such cases. To employ this method, it is first necessary to specify relations that govern flow parameters on both sides of the discontinuity (1.3).

Let us assume that at the discontinuity (1.3), the laws of conservation of mass and local momentum

$$h_t + q_x = 0, \qquad v_t + (v^2/2 + z)_x = 0, \tag{2.6}$$

and, hence, the law of total energy conservation are satisfied [7]. From relation (2.6), it follows that the flow rate and the Bernoulli constant are constant at the step:

$$[q] = 0, \qquad [v^2/2 + z] = 0. \tag{2.7}$$

This implies that for evolution [2, 12] of a fixed discontinuity over a step, it is necessary that two characteristics come to the discontinuity and two characteristics go strictly out from it (the characteristics propagating at zero velocity along the discontinuity line (1.3) are included in the number of incoming characteristics and are not included in the number of strictly outgoing characteristics). In [7], it is shown that within the framework of such evolutionary-steady discontinuous solutions, the flow depth $H$ and velocity $V$ ahead of the step are uniquely determined from their values $h$ and $v$ at the step. In this case, the bed level discontinuity (1.3) in the shallow-water model corresponds to a transition zone $[-\varepsilon, \varepsilon]$ of a rather fast but smooth and monotonic change in the bed level of an actual channel (see [7, Secs. 4 and 5]).

**3. Conservation of the Monotonicity of the Function $v(h)$ with Passage through a Bottom Step.** We consider a one-parameter family of steady discontinuous solutions with depths and velocities $h$ and $v(h)$ at the step and $H(h)$ and $V(h)$ ahead of it. We elucidate under what conditions the monotonicity of the function $v(h)$ leads to the monotonicity of the functions $H(h)$ and $V(h)$ and, hence, the monotonicity of the function $\tilde{V}(H) = V(h(H))$ [$h(H)$ is a function that is the reverse of $H(h)$].

**Theorem 1.** *If a function $v(h)$ satisfying the conditions $v > 0$ and $v_h > 0$ takes values in the subcritical and critical flow region $v \leqslant \sqrt{h}$ (in the supercritical flow region, $v > \sqrt{h}$), then the functions $H(h)$ and $V(h)$ corresponding to it satisfy the inequalities*

$$H_h > 0, \quad V > 0, \quad V_h > 0 \quad \Rightarrow \quad \tilde{V}_H > 0,$$

*and the function $V = \tilde{V}(H)$ takes values in the subcritical flow region $V < \sqrt{H}$ (in the supercritical flow region, $V > \sqrt{H}$).*

**Proof.** The proof of Theorem 1 starts with proving the inequality $H_h > 0$. The Hugoniot conditions (2.7) lead to the relation

$$J(H, q) = J(h, q) + \delta, \tag{3.1}$$

in which

$$J(H, q) = q^2/(2H^2) + H, \qquad J(h, q) = q^2/(2h^2) + h, \qquad q = hv = HV.$$

Since the total differential of Eq. (3.1) can be written as

$$\alpha_1 \, dH = \alpha_0 \, dh + (v/h - V/H) \, dq,$$

where

$$\alpha_0 = J_h(h, q) = 1 - v^2/h, \qquad \alpha_1 = J_H(H, q) = 1 - V^2/H, \tag{3.2}$$

then, taking into account that the flow on the left of the discontinuity (1.3) cannot be critical [7] and, hence, $\alpha_1 \neq 0$, we obtain

$$H_h = \frac{1}{\alpha_1}\Big(\alpha_0 + \Big(\frac{v}{h} - \frac{V}{H}\Big) q_h\Big) = \frac{1}{\alpha_1}\Big(\alpha_0 + \frac{q(H^2 - h^2)}{h^2 H^2} \, q_h\Big), \tag{3.3}$$

where

$$q_h = (hv)_h = v + h v_h. \tag{3.4}$$

By the condition of Theorem 1, $v > 0$ and $v_h > 0$,; therefore, taking into account (3.4), we have $q > 0$ and $q_h > 0$. Consequently, for the permissible flow configurations at the step obtained in [7], for the first of which,

$$H > h + \delta, \qquad \alpha_1 > 0, \qquad \alpha_0 \geqslant 0, \tag{3.5}$$

and for the second,

$$H < h, \qquad \alpha_1 < 0, \qquad \alpha_0 \leqslant 0, \tag{3.6}$$

from (3.3), we have $H_h > 0$.

To determine the sign of the derivative $V_h$, we write it as

$$V_h = (q/H)_h = (H q_h - q H_h)/H^2.$$

From this, taking into account (3.3) and then (3.2) and (3.4), we obtain

$$V_h = ((\alpha_1 H + V^2 - v^2) q_h - \alpha_0 q)/(\alpha_1 H^2)$$

$$= ((H - v^2) q_h - \alpha_0 q)/(\alpha_1 H^2) = ((H - h) q_h + \alpha_0 h^2 v_h)/(\alpha_1 H^2). \tag{3.7}$$

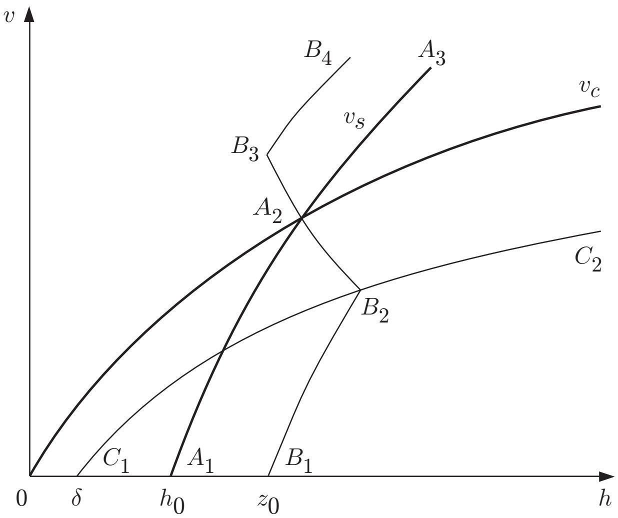


Fig. 1

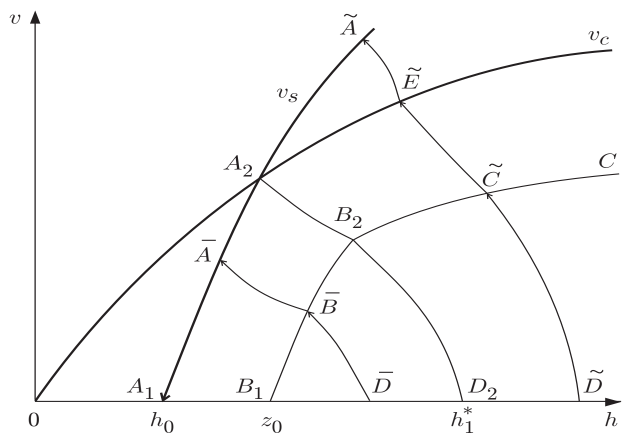


Fig. 2

Since $v_h > 0$ and $q_h > 0$, from formula (3.7) it follows that $V_h > 0$ both under conditions (3.5), where the values of the function $\tilde{V}(H)$ belong to the subcritical flow region and under conditions (3.6), where the values of the function $\tilde{V}(H)$ belong to the supercritical flow region. Thus, Theorem 1 is proved.

As is shown in [7], each shock $s$-adiabat

$$v_s(h, h_0, 0) = a(h, h_0) = \sqrt{(h + h_0)/(2hh_0)}(h - h_0) \tag{3.8}$$

issuing from the point $h_0$ on the axis $h$ (Fig. 1) intersects the critical flow curve $v = \sqrt{h}$ at a single point $A_2 = (x^* h_0, \sqrt{x^* h_0})$, where

$$x^* = 1 + \frac{4}{\sqrt{3}} \cos\Big(\frac{1}{3} \arccos \frac{3\sqrt{3}}{8}\Big) \approx 3.214 \tag{3.9}$$

is the maximal root of the cubic equation $x^3 - 3x^2 - x + 1 = 0$. Therefore, by virtue of Theorem 1, upon passage through the step, the subcritical part $A_1A_2$ of this adiabat is mapped into the monotonic curve shown in Fig. 1 by the curve $B_1B_2$, lying in the subcritical flow region, and its supercritical part $A_2A_3$ is mapped into the monotonic curve shown in Fig. 1 by the curve $B_3B_4$ lying in the supercritical flow region. In this case, the transition $A_1A_2 \to B_1B_2$ corresponds to configuration (3.5) and the transition $A_2A_3 \to B_3B_4$, to configuration (3.6). The critical flow line $v = v_c(h) = \sqrt{h}$ is mapped into the monotonic curve shown in Fig. 1 by the curve $C_1B_2C_2$, located in the subcritical flow region.

**4. Solvability of the Discontinuity Decay Problem (1.1)–(1.4).** The one-valued solvability of the discontinuity decay problem (1.1)–(1.4) under conditions (2.7) at the bottom step follows from a monotonic decrease in the wave $r$-adiabat

$$v_r(h, h_1, 0) = -a(h, h_1) = 2(\sqrt{h_1} - \sqrt{h}) \tag{4.1}$$

issuing from the point $h_1 > z_0 = h_0 + \delta$ on the axis $h$ (Fig. 2) and a monotonic increase in the piecewise smooth curve $B_1B_2C$, whose segment $B_1B_2$ is the image of the subcritical part $A_1A_2$ of the shock $s$-adiabat (3.8) and the segment $B_2C$ is the image of the part of the critical flow line $v = \sqrt{h}$ located to the right of the point $A_2$, i.e., at $h > h_0^* = x^* h_0$.

We use $h_1^*$ to denote the point on the axis $h$ which is the origin of the $r$-adiabat (4.1) going to the point $B_2$ of inflection of the curve $B_1B_2C$. Then, for $h_1 \in (z_0, h_1^*]$, the $r$-adiabat (4.1) intersects this curve on the segment $B_1B_2$, resulting in the formation of the flow pattern shown in Fig. 5a in [7]. In this case, the depression wave $R$ corresponds to the segment $\bar{D}\bar{B}$ of the wave $r$-adiabat (4.1), the discontinuity $L$ at the step corresponds to the shock transition $\bar{B}\bar{A}$ along the hyperbola $v = \bar{q}/h$, and the discontinuous wave $S$ corresponds to the segment $\bar{A}A_1$ of the shock $s$-adiabat (3.8). At $h_1 > h_1^*$, the $r$-adiabat (4.1) intersects the curve $B_1B_2C$ on the segment $B_2C$, which results in the formation of the flow pattern shown in Fig. 5b in [7]. In this case, the depression wave $R$ located on the left of the step corresponds to the segment $\tilde{D}\tilde{C}$ of the wave $r$-adiabat (4.1), the discontinuity $L$ corresponds to the shock transition $\tilde{C}\tilde{E}$, the depression wave $R_1$ located on the right of the step corresponds to the segment $\tilde{E}\tilde{A}$ of the wave $r$-adiabat (2.2) issuing from the point $\tilde{E}$, and the discontinuous wave $S$ corresponds to the segment $\tilde{A}A_1$ of the shock $s$-adiabat (3.8). In this case, the flow on the right of the step is obtained by solving the classical discontinuity decay problem with initial data lying at the points $A_1$ and $\tilde{E}$ in Fig. 2.

The point $h_1^*$ separating the two indicated flows is determined as follows. First, using the formulas

$$h_2 = x^* h_0, \qquad v_2 = \sqrt{h_2} = \sqrt{x^* h_0}, \tag{4.2}$$

where $x^*$ is given by relation (3.9), we obtain the coordinates of the point $A_2$ of intersection of the shock $s$-adiabat (3.8) and the critical flow line $v = \sqrt{h}$. Then, using the formulas

$$h_3 = a\Big(1 + 2\cos\Big(\frac{1}{3}\arccos\frac{a^3 - q^2/4}{a^3}\Big)\Big), \qquad v_3 = \frac{q}{h_3}, \tag{4.3}$$

where

$$a = (v_2^2 + 2z_2)/6, \qquad q = h_2 v_2, \qquad z_2 = h_2 + \delta \tag{4.4}$$

(these formulas are obtained in [7]), we find the coordinates of the point $B_2$ into which the point $A_2$ is mapped upon passage through the step. Finally, from the formula following from (4.1)

$$h_1^* = (\sqrt{h_3} + v_3/2)^2 \tag{4.5}$$

we calculate the coordinate of the point $D_2$ which is the origin of the wave $r$-adiabat (4.1) passing through the point $B_2$.

**5. Method of Adiabats Using the Flow Rate $q$ and Bernoulli Constant $Q$ as Variables.** In the method of adiabats used for the equations of gas dynamics [11], the pressure $p$ and velocity $v$ are variables which are continuous at a discontinuity. Similarly, in solving the generalized discontinuity decay problem (1.2)–(1.4) for the shallow water equations (1.1), the method of adiabats is conveniently applied using the flow rate $q$ and Bernoulli constant $Q = v^2/2 + z$ as variables which remain continuous at the discontinuity over the step (1.3). A similar approach is used in [8] in solving the problem of decay of a boundary discontinuity over an even bottom. To apply this approach to the solution of problem (1.1)–(1.4), one needs to show that the shock $s$-adiabat (3.8) and the wave $r$-adiabat (4.1) remain monotonic when written as the functions

$$q = q_s(Q, z_0), \qquad q = q_r(Q, z_1), \tag{5.1}$$

where $z_0$ and $z_1$ are the initial levels included in formula (1.2).

**Theorem 2.** *Each positive monotonically increasing function $v(h)$ going out of the points $h_0 \geqslant 0$ on the axis $h$ can be written as a function $q = \tilde{q}(Q)$ which is strictly monotonically increasing for $Q > z_0 = h_0 + \delta$.*

**Proof.** If the function $v(h)$ satisfies the condition

$$v > 0, \qquad v_h > 0 \qquad \forall h > h_0 \geqslant 0, \qquad v(h_0) = 0,$$

the corresponding functions

$$q(h) = hv(h), \qquad Q(h) = v^2(h)/2 + h + \delta$$

satisfy the conditions

$$q > 0, \qquad q_h > 0, \qquad Q_h > 0 \qquad \forall h > h_0.$$

This implies that for $Q \geqslant z_0 = h_0 + \delta > 0$, a function $h(Q)$ that is the reverse of $Q(h)$ is defined and, hence, a function $\tilde{q}(Q) = q(h(Q))$ is defined such that

$$\tilde{q} > 0, \quad \tilde{q}_Q = q_h/Q_h = (v + hv_h)/(1 + vv_h) > 0 \quad \forall Q > z_0, \quad \tilde{q}(z_0) = 0.$$

Theorem 2 is proved.

From Theorem 2, it follows that the shock $s$-adiabat (3.8) and the critical flow line $v = \sqrt{h}$ are plotted on the plane of the variables $(Q, q)$ as curves of strictly monotonically increasing functions $q = q_s(Q)$ and $q = q_c(Q)$, which are presented in Fig. 3.

In contrast to the monotonically increasing functions, the monotonically decreasing functions $v(h)$ generally lose the monotonicity property when written in the variables $Q$ and $q$. For example, the monotonically decreasing linear function $v = 1 - h$ written as

$$q = h(1 - h), \qquad h = \sqrt{2(Q - \delta) - 1}$$

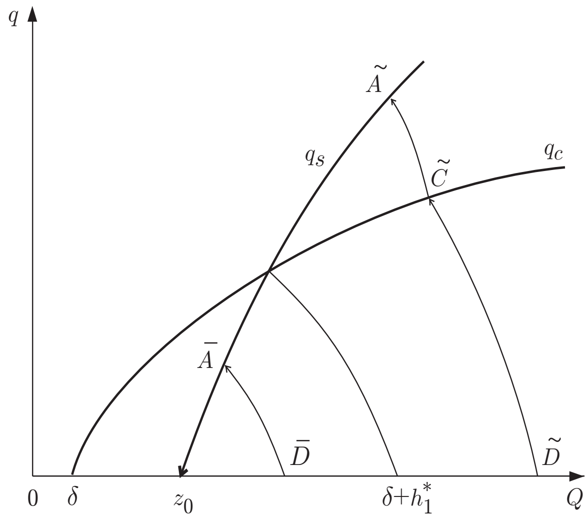


Fig. 3

becomes monotonically increasing for $h \in (0, 1/2) \Leftrightarrow Q \in (\delta + 1/2, \delta + 5/8)$ and remains monotonically decreasing for $h \in (1/2, 1) \Leftrightarrow Q \in (\delta + 5/8, \delta + 1)$. In spite of this, for the wave $r$-adiabats (4.1), the following formula holds:

$$\tilde{q}_Q = \frac{q_h}{Q_h} = \frac{1 + v_r(v_r)_h}{v_r + h(v_r)_h} = \frac{1 - 2(\sqrt{h_1} - \sqrt{h})/\sqrt{h}}{2(\sqrt{h_1} - \sqrt{h}) - h/\sqrt{h}} = -\frac{1}{\sqrt{h}} < 0.$$

By virtue of this, the adiabat remains monotonically decreasing when written in the variables $Q$ and $q$.

From the monotonicity of adiabats (5.1), it follows that the one-valued solvability of the discontinuity decay problem (1.1)–(1.4) can be shown in the variables $Q$ and $q$. Depending on whether the adiabats (5.1) intersect in the subcritical or supercritical region (points $\bar{A}$ and $\tilde{A}$ in Fig. 3) the flow that arises has the form shown in [7, Fig. 5a or Fig. 5b]. The shock transitions over the step shown by the curves $\bar{B}\bar{A}$ and $\tilde{C}\tilde{E}$ in Fig. 2 are concentrated at the points $\bar{A}$ and $\tilde{C}$ in Fig. 3.

**6. Discontinuity Decay Problem (1.1)–(1.4) with Energy Loss at the Bottom Step.** As noted in [7], the over shallow- water model with conditions (2.7) at the discontinuity (1.3), leading to conservation of the total flow energy at the step, generally describes actual flows for which the discontinuity (1.3) simulates a transition zone of rather fast but smooth and monotonic change in the bed level of an actual channel. At the same time, it is of interest to analyze the possibility of using the shallow water model to describe flows over bed level discontinuities in actual channels in the case where the total energy of the flow at the step (1.3) is not conserved. In this case, only the law of conservation of mass is satisfied at the discontinuity (1.3) $[q] = 0$. To close the conditions at such a discontinuity, it does not suffice to specify characteristics that arrive at it; one needs to specify one more scalar relation or the presence of the third incoming characteristic.

Let us first assume that two characteristics arrive at the discontinuity (1.3), where $[Q] \neq 0$,; then it is necessary to specify one more scalar relation at the discontinuity. In this situation, in studies of gas-dynamic flows in a pipe with a sudden change in cross-sectional area [9, 10], the nondivergent equation of total momentum, was used as the additional relation at the cross-section discontinuity, in which the response of the wall between pipelines of different diameters was taken into account for various physical reasons (beyond the scope of the purely one-dimensional model).

We apply a different approach in which the second relation at the is the modified energy balance equation (3.1)

$$\sigma J(H, q) = J(h, q) + \delta, \tag{6.1}$$

whose left side contains the heuristic parameter $\sigma \in (0, 1]$ representing the part of the total flow energy conserved upon transition through the discontinuity (1.3). An advantage of this approach is that it is a direct generalization of the above case with conservation of total energy at the step. The specific value of the parameter $\sigma$ should be chosen by analyzing the fine structure of the flow in the neighborhood of an actual step, which is impossible within the framework of the shallow-water model. We assume, therefore, that the parameter $\sigma$ included in condition (6.1)

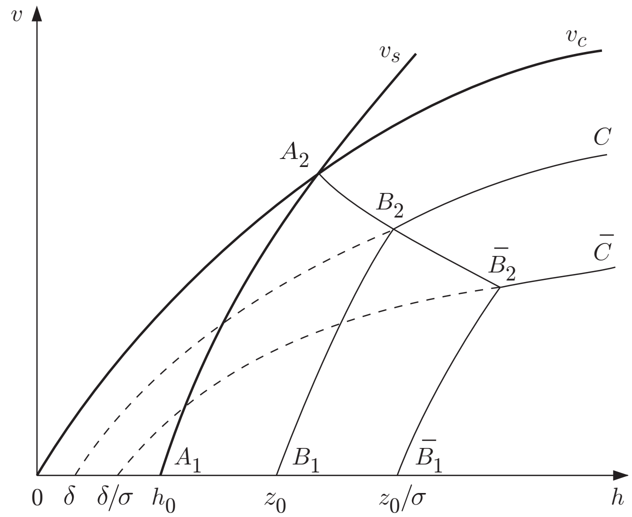


Fig. 4

is known beforehand; bearing this in mind we analyze the solvability of the generalized discontinuity decay problem (1.1)–(1.4).

Because (6.1) leads to

$$q^2 = 2h^2H^2(\sigma H - h - \delta)/(H^2 - \sigma h^2)$$

it follows that at the discontinuity (1.3), only those flows are possible for which

$$(\sigma H - h - \delta)(H - \sqrt{\sigma}h) > 0.$$

This inequality distinguishes two permissible flow configurations over the step, for the first of which,

$$H > (h + \delta)/\sigma, \qquad V^2 < H, \tag{6.2}$$

and for the second,

$$H < \sqrt{\sigma}h, \qquad V^2 > H. \tag{6.3}$$

Here $V^2 = 2h^2(\sigma H - h - \delta)/(H^2 - \sigma h^2)$ is the square of the flow velocity ahead of the step.

Just as in [7], it is possible to show that within the framework of steady discontinuous solutions for which two characteristics arrives at the discontinuity (1.3), Eq. (6.1) is uniquely solvable for $H$ at $h > 0$ and for $h$ at $H$, such that the inequality $J(H, q) = q^2/(2H^2) + H > \delta/\sigma$ is satisfied. If $q > 0$, the flow at the step is subcritical or critical ($v^2 \leqslant h$) in the case of conditions (6.2) and is supercritical ($v^2 > h$) in the case of conditions (6.3). In view of this, for steady flows over a bottom step, Theorem 1 is valid. To prove this theorem for $\sigma < 1$, it suffices to note that in the case of the energy relation (6.1) formulas (3.3) and (3.7) become

$$H_h = \frac{1}{\sigma\alpha_1}\Big(\alpha_0 + \frac{q(H^2 - \sigma h^2)}{h^2H^2}\, q_h\Big), \qquad V_h = \frac{(\sigma H - h)q_h + \alpha_0 h^2 v_h}{\sigma\alpha_1 H^2}.$$

By virtue of this, the positiveness of the derivatives $H_h$ and $V_h$ (as for $\sigma = 1$) directly follows from the fact that the inequalities $\alpha_1 > 0$ and $\alpha_0 \geqslant 0$ hold under condition (6.2) and the inverse inequalities $\alpha_1 < 0$ and $\alpha_0 \leqslant 0$ hold under conditions (6.3). From this it follows that the subcritical part $A_1A_2$ of the shock $s$-adiabat (3.8) and the part of the critical flow line $v = \sqrt{h}$ located to the right of the point $A_2$ (see Figs. 1 and 4) are transferred by relation (6.1), where $\sigma < 1$, into the monotonically increasing curves shown in Fig. 4 by the curve $\bar{B}_1\bar{B}_2$, lying to the right of the curve $B_1B_2$, and the curve $\bar{B}_2\bar{C}$, located below the curve $B_2C$.

If the inequality

$$h_1 > (h_0 + \delta)/\sigma \quad \Longleftrightarrow \quad z_1 > z_0/\sigma \tag{6.4}$$

holds, the one-valued solvability of the discontinuity decay problem (1.1)–(1.4) subject to condition (6.1) at the bottom step follows from a monotonic increase of the piecewise smooth curve $\bar{B}_1\bar{B}_2\bar{C}$ in Fig. 4 and a monotonic decrease in the wave $r$-adiabat (4.1). If

$$h_0 + \delta < h_1 \leqslant (h_0 + \delta)/\sigma \quad \Longleftrightarrow \quad z_0 < z_1 \leqslant z_0/\sigma, \tag{6.5}$$

this problem has no solution.

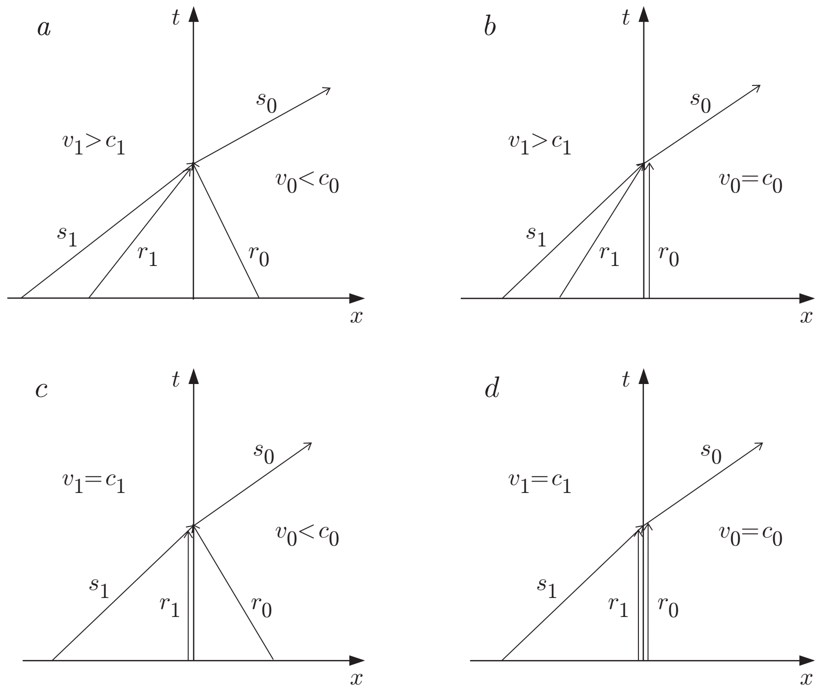


Fig. 5

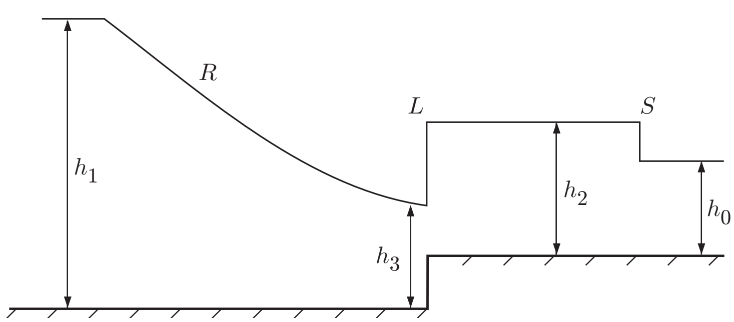


Fig. 6

To obtain the a plot of the adiabats corresponding to condition (6.4), the curve $B_1B_2C$ in Fig. 2 obtained for $\sigma = 1$, should be replaced by the curve $\bar{B}_1\bar{B}_2\bar{C}$ (see Fig. 4) obtained for $\sigma < 1$. From this it follows that as in the case of $\sigma = 1$, for $\sigma < 1$, the discontinuity decay problem (1.1)–(1.4) admits two types of solutions (see [7, Fig. 5a and b]). The coordinate $h_1^*(\sigma)$ of the point $D_2$ in Fig. 2 separating these two types of flows is calculated from formulas (4.2)–(4.5), in which for $\sigma < 1$, the parameter $a$ included in (4.3) should be written as $a = (v_2^2+2z_2)/(6\sigma^2)$; in this case, the function $h_1^*(\sigma)$ is monotonically decreasing and

$$\lim_{\sigma\to 0} h_1^*(\sigma) = +\infty.$$

**7. Steady Flows in the Case of Three Characteristics Arriving at a Discontinuity Line over a Bottom Step.** If three characteristics arrive at a discontinuity line that arises over a step, the condition of continuity of the flow rate $[q] = 0$ is sufficient to close the conditions at such a discontinuity; furthermore, these conditions uniquely define the part of the total flow energy lost upon passage through the step. We note that a similar situation arises in the solution of the problem of decay of a boundary discontinuity over an even bottom [8].

For $q > 0$, in order that three characteristics arrive at the discontinuity (1.3), it is necessary that the flow $(H, V)$ ahead of the step be supercritical or critical ($V \geqslant \sqrt{H}$) and the flow $(h, v)$ at the step be subcritical or critical ($v \leqslant \sqrt{h}$). The fields of characteristics corresponding to these cases are shown in Fig. 5. In the solution of the discontinuity decay problem (1.1)–(1.4), an $r$-depression wave (denoted in Fig. 6 by letter $R$) propagates to the left of the bottom step. The flow on the right of this wave can be subcritical or critical; the critical flow behind the

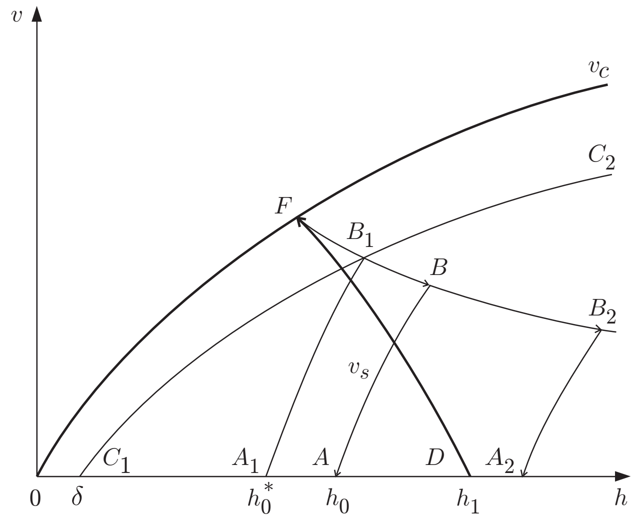


Fig. 7

wave $R$ forms only when its right boundary is adjacent to the discontinuity $L$ over the step, forming with it a single wavy $LR$ jump. Thus, in the case considered, the flow ahead of the step is critical. Since its parameters $h_3$ and $v_3$ are uniquely determined as the coordinates of the point $F$ (Fig. 7) at which the critical flow line $v = v_c(h) = \sqrt{h}$ intersects the wave $r$-adiabat (4.1), they can be calculated from the formulas

$$h_3 = (4/9)h_1, \qquad v_3 = \sqrt{h_3} = (2/3)\sqrt{h_1}.$$

Since the flow at the step cannot be supercritical, its parameters $h_2$ and $v_2$ belong to the part of the hyperbola

$$v = q_3/h, \qquad q_3 = h_3 v_3 \tag{7.1}$$

that issues from the point $F$ in Fig. 7 into the subcritical flow region; for the energy stability of the discontinuity $L$, related to the loss of total energy at it, the point corresponding to these parameters should lie on hyperbola (7.1) to the right of the point $B_1$, at which it intersects the curve $C_1C_2$, which is the image of the critical flow line $v_c(h)$ upon transition through the discontinuity $L$ subject to the condition of total energy conservation $[Q] = 0$. This implies that the flow $(h_2, v_2)$ is subcritical, and, therefore, in the case considered, the steady flows at the step correspond to the field of characteristics shown in Fig. 5c. In this case, the coordinates $(h^*, v^*)$ of the point $B_1$ are determined from the formulas

$$h^* = a\Big(1 + 2\cos\Big(\frac{1}{3}\arccos\frac{a^3 - q^2/4}{a^3}\Big)\Big), \qquad v^* = \frac{q}{h^*},$$

where $a = (v_3^2 + 2(h_3 + \delta))/6$ and $q = h_3 v_3$.

Since the flow $h_2, v_2$ is subcritical, only a discontinuous $s$-wave (denoted by letter $S$ in Fig. 6) can propagate to the right of the step. The parameters $h_2$ and $v_2$ of the constant flow between the discontinuity $L$ and the discontinuous wave $S$ are uniquely determined as the coordinates of the point $B$ of intersection (Fig. 7) of hyperbola (7.1) and the shock $s$-adiabat (3.8) issuing from the point $A$ on the axis $h$. In this case, the coordinate $h_0$ of the point $A$ should satisfy the condition

$$h_0 \in (h_0^*, h_1 - \delta), \tag{7.2}$$

where $h_0^* = x^* h^*$ is the coordinate of the point $A_1$, which is the origin of the shock $s$-adiabat $v_s(h, h_0^*, 0)$ passing through the point $B_1$ (Fig. 7). The value $x^* \in (0, 1)$ is a root of the cubic equation

$$x^3 - x^2 - (2f^2 + 1)x + 1 = 0$$

and is calculated from Cardano's formula

$$x^* = \frac{1}{3}\Big(2p\cos\Big(\frac{1}{3}\Big(2\pi - \arccos\frac{9f^2 - 8}{p^3}\Big)\Big) + 1\Big),$$

where $p = \sqrt{6f^2 + 4}$ and $f = v^*/\sqrt{h^*}$. Condition (7.2) imposes constraint on the initial depths $h_0$ and $h_1$ and the step height $\delta$, for which the problem (1.1)–(1.4) admits the steady solutions shown in Fig. 6.

**Conclusions.** The present study shows the one-valued solvability of the generalized problem of discontinuity decay (1.1)–(1.4) under the assumption that the total energy of the flow at the bottom step is conserved. For experimental verification of the self-similar solutions obtained in this case, V. I. Bukreev and A. V. Gusev, researchers from the Laboratory of Experimental Applied Hydrodynamics of the Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, performed a series of experiments in which the step (1.3) was modeled by a segment of monotonic linear rise in bed level and the initial level discontinuity (1.2) was produced by a shield located ahead of this rise. The experiments showed good agreement between theoretical and experimental data on the speed of propagation of discontinuous $s$-waves and the asymptotic depth behind the wave front. These experimental results will be published.

In the present work, we constructed two classes of solutions of problem (1.1)–(1.4) for which the total energy of the flow over the step is lost. For solutions of the first class (see Sec. 6), for which two characteristics arrive at the discontinuity $L$ over the step, closure of the model requires introducing a heuristic parameter $\sigma$ that specifies the part of the total flow energy conserved with passage through the step. The solutions obtained for this case are qualitatively similar to those obtained for the case of conservation of total energy at the discontinuity (1.3); in particular, in the case of flow onto the step, the water line always drops (see [7, Fig. 5]).

In contrast to solutions of the first class, solutions of the second class (see Sec. 7), for which three characteristics arrive at the discontinuity line $L$, are uniquely determined within the framework of the shallow- water model without using any heuristic parameters. These solutions differ qualitatively from the solutions with conservation of total energy at the discontinuity (1.3) because they always involve an increase in the level of the fluid flowing onto the step (see Fig. 6). Formally, both classes of solutions can exist for the same initial data (1.2)–(1.4). Therefore, for their separation in laboratory experiments, one needs to study the fine flow structure in the neighborhood of an actual step when simulating the dam-break process by fast removal of the shield separating liquids of different levels.

It can be assumed that if the shield is sufficiently thin, its removal gives rise to flows of the first class. If the shield has finite width on the axis $x$ and, being adjacent to the step, is completely located in the region $x < 0$, fast removal of the shield can give rise to flows of the second class. For this, it is necessary that the liquid velocity on the right boundary of the depression wave propagating over background $z_1$ reach the critical value when filling the hole that arises on the left of the step.

It should be noted that, formally, steady solutions of the second class can exist not only at $z_1 > z_0$ but also at $z_1 < z_0$ and even at $h_1 < h_0$. In Fig. 7, the adiabats of one of such flows are shown by the curves $DF$ and $B_2A_2$ If such paradoxical (in a sense) flows occur, decay of the discontinuity (1.2)–(1.4) will cause the liquid to propagate toward the higher rather than the lower initial level. Corresponding laboratory experiments are required to verify the existence of such flows.

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